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# Observability Analysis of Six-Degree-of-Freedom Configuration Determination Using Vector Observations

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**An observability analysis of the six-degree-of-freedom attitude and position determination problem using line-of-sight observations is shown. This analysis involves decompositions of the associated error covariance matrix, derived from maximum likelihood, for a number of cases ranging from one vector observation to three or more vector observations. The covariance matrix is shown to be singular when one or two vector observations are used, leading to an unobservable system. For the one-vector case, the observable quantities involve a combination of both attitude and position information that cannot be decoupled. For the two-vector case, the covariance matrix has rank four, but only one axis of attitude and one axis of position is fully observable, with the other two observable quantities involving coupled attitude/position information. When three or more vector observations are present, the covariance matrix has full rank, except for some special cases that are derived. This observability analysis is useful for the design and analysis of estimators using line-of-sight vector observations.**

## Introduction

BOTH the attitude and the position of a vehicle can be determined from line-of-sight (LOS) vector observations. One mechanism to accomplish this task involves a vision navigation (VISNAV) system based on position-sensing diodes in the focal plane of a camera, which allows the inherent centroiding of a light-emitting diode beacon's incident light.[1] Other mechanisms may involve camera image measurements or laser reflector LOS measurements. The fundamental approach used to determine the attitude and position from LOS observations involves an object to image projective transformation, achieved through the collinearity equations.[2] These equations involve the angle of the body from the sensor boresight in two mutually orthogonal planes, which can be reconstructed into unit vector form. The most common approach to determine attitude and position using the collinearity equations involves a Gaussian least squares differential correction (GLSDC) process, whereas a new estimation approach has been presented in Ref. 3 based on a predictive filter for nonlinear systems.

Determining attitude from LOS observations commonly involves finding a proper orthogonal matrix that minimizes the scalar weighted norm error between sets of $3 \times 1$ body vector observations and $3 \times 1$ known reference vectors mapped (via the attitude matrix) into the body frame. This is known as Wahba's problem.[4] If the reference vectors are known, then at least two noncollinear unit vector observations are required to determine the attitude. Many methods have been developed that solve this problem efficiently and accurately.[5,6] Determining the position from LOS observations involves triangulation from known reference base points. If the attitude is known, then at least two noncollinear unit vector observations are required to establish a three-dimensional position. Determining both attitude and position from LOS observations is more complex because more than two noncollinear unit vector observations are required (as will be demonstrated in this paper), and, unlike Wahba's problem,[4] the unknown attitude and position are interlaced in a highly nonlinear fashion.

In this paper, an analysis is performed to study the observability of the coupled attitude and position determination problem from vector observations. In Ref. 3, an initial study was performed for the two-vector observation case, which showed that only one axis of attitude and one axis of position information can be determined for this case. Furthermore, an observability analysis using two-vector observations indicates that the beacon that is closest to the target provides the most attitude information, but has the least position information, and that the beacon that is farthest from the target provides the most position information, but has the least attitude information. This paper extends this initial result for the one- and three or more vector observation cases and also more fully quantifies the two vector observation case.

The organization of this paper is as follows. First, a review of the collinearity equations is shown. Then, a generalized loss function derived from maximum likelihood for attitude and position determination is given. Next, the optimal estimate covariance is derived, which gives the Cramér–Rao lower bound. Then, an observability analysis is shown for cases involving one to three or more vector observations. This analysis is performed using an eigenvalue/eigenvector decomposition of the information matrix, that is, the inverse of the covariance matrix. Finally, the trace and eigenvalues of the covariance matrix are studied.

## Collinearity Equations and Covariance

In this section, the collinearity equations for attitude and position determination are shown. First, the observation model is reviewed. Then, the estimate (attitude and position) covariance matrix is derived using maximum likelihood.

### Collinearity Equations

Photogrammetry is the technique of measuring objects (two or three dimensions) from photographic images or LOS measurements. Photogrammetry can generally be divided into two categories: far-range photogrammetry with camera distance settings to infinity (commonly used in star cameras[7]) and close-range photogrammetry with camera distance settings to finite values. In general, close-range photogrammetry can be used to determine both the position and attitude of an object, whereas far-range photogrammetry can only be used to determine attitude. The relationship between the position/attitude and the observations used in photogrammetry involves a

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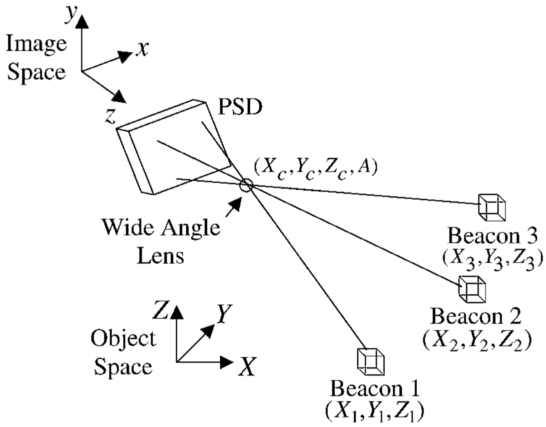


**Fig. 1 VISNAV system.**

set of collinearity equations, which are reviewed in this section. Figure 1 shows a schematic of the typical quantities involved in basic photogrammetry from LOS measurements, derived from light beacons in this case. If we choose the $z$ axis of the sensor coordinate system to be directed outward along the boresight, then, given object space $(X, Y, Z)$ and image space $(x, y, z)$ coordinate frames (see Fig. 1), the ideal object to image space projective transformation (noiseless) can be written as follows[8]:

$$x_i = -f\frac{A_{11}(X_i - X_c) + A_{12}(Y_i - Y_c) + A_{13}(Z_i - Z_c)}{A_{31}(X_i - X_c) + A_{32}(Y_i - Y_c) + A_{33}(Z_i - Z_c)} \qquad i = 1, 2, \ldots, N \tag{1a}$$

$$y_i = -f\frac{A_{21}(X_i - X_c) + A_{22}(Y_i - Y_c) + A_{23}(Z_i - Z_c)}{A_{31}(X_i - X_c) + A_{32}(Y_i - Y_c) + A_{33}(Z_i - Z_c)} \qquad i = 1, 2, \ldots, N \tag{1b}$$

where $N$ is the total number of observations, $x_i$ and $y_i$ are the image space observations for the $i$th LOS, $X_i$, $Y_i$, and $Z_i$ are the known object space locations of the $i$th beacon, $X_c$, $Y_c$, and $Z_c$ are the unknown object space locations of the sensor, $f$ is the known focal length, and $A_{jk}$ are the unknown coefficients of the attitude matrix $A$ associated with the orientation from the object plane to the image plane. The goal of the inverse problem is, given observations $(x_i, y_i)$ and object space locations $(X_i, Y_i, Z_i)$, for $i = 1, 2, \ldots, N$, to determine the attitude $A$ and position $(X_c, Y_c, Z_c)$. This can be accomplished by using a GLSDC process or by other methods.[3]

The observation can be reconstructed in unit vector form as

$$\boldsymbol{b}_i = A\boldsymbol{r}_i, \qquad i = 1, 2, \ldots, N \tag{2}$$

where

$$\boldsymbol{b}_i \equiv \frac{1}{\sqrt{f^2 + x_i^2 + y_i^2}}\begin{bmatrix} -x_i \\ -y_i \\ f \end{bmatrix} \tag{3a}$$

$$\boldsymbol{r}_i \equiv \frac{1}{\sqrt{(X_i - X_c)^2 + (Y_i - Y_c)^2 + (Z_i - Z_c)^2}}\begin{bmatrix} X_i - X_c \\ Y_i - Y_c \\ Z_i - Z_c \end{bmatrix} \tag{3b}$$

When measurement noise is present, Shuster[5] has shown that nearly all of the probability of the errors is concentrated on a very small area about the direction of $A\boldsymbol{r}_i$, so that the sphere containing that point can be approximated by a tangent plane, characterized by

$$\tilde{\boldsymbol{b}}_i = A\boldsymbol{r}_i + \boldsymbol{v}_i, \qquad \boldsymbol{v}_i^T A\boldsymbol{r}_i = 0 \tag{4}$$

where $\tilde{\boldsymbol{b}}_i$ denotes the $i$th measurement and the sensor error $\boldsymbol{v}_i$ is approximately Gaussian, which satisfies

$$E\{\boldsymbol{v}_i\} = \boldsymbol{0} \tag{5a}$$

$$E\left\{\boldsymbol{v}_i\boldsymbol{v}_i^T\right\} = \sigma_i^2\left[I - (A\boldsymbol{r}_i)(A\boldsymbol{r}_i)^T\right] \tag{5b}$$

and $E\{\}$ denotes expectation. Equation (5b) makes the small field-of-view assumption of Ref. 5; however, for a large field-of-view lens with significant radial distortion, this covariance model should be modified appropriately.

**Maximum Likelihood Estimation and Covariance**

Attitude and position determination using LOS measurements involves finding estimates of the proper orthogonal matrix $A$ and position vector $\boldsymbol{p} \equiv [X_c \;\; Y_c \;\; Z_c]^T$ that minimize the following loss function:

$$J(\hat{A}, \hat{\boldsymbol{p}}) = \frac{1}{2}\sum_{i=1}^{N}\sigma_i^{-2}\|\tilde{\boldsymbol{b}}_i - \hat{A}\hat{\boldsymbol{r}}_i\|^2 \tag{6}$$

where the carat denotes estimate. An estimate error covariance can be derived from the loss function in Eq. (6). This is accomplished by using results from maximum likelihood estimation.[5,9] The Fisher information matrix for a parameter vector $\boldsymbol{x}$ is given by

$$F_{xx} = E\left\{\frac{\partial}{\partial\boldsymbol{x}\partial\boldsymbol{x}^T}J(\boldsymbol{x})\right\}_{\boldsymbol{x}_{\text{true}}} \tag{7}$$

where $J(\boldsymbol{x})$ is the negative log-likelihood function, which is the loss function in this case (neglecting terms independent of $A$ and $\boldsymbol{p}$). Asymptotically, the Fisher information matrix tends to the inverse of the estimate error covariance so that $\lim_{N\to\infty} F_{xx} = P^{-1}$. The true attitude matrix is approximated by

$$A = e^{-[\delta\alpha\times]}\hat{A} \approx (I_{3\times 3} - [\boldsymbol{\delta\alpha}\times])\hat{A} \tag{8}$$

where $\boldsymbol{\delta\alpha}$ represents a small angle error and $I_{3\times 3}$ is a $3\times 3$ identity matrix. The $3\times 3$ matrix $[\boldsymbol{\delta\alpha}\times]$ is referred to as a cross-product matrix because $\boldsymbol{a}\times\boldsymbol{b} = [\boldsymbol{a}\times]\boldsymbol{b}$, with

$$[\boldsymbol{a}\times] \equiv \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \tag{9}$$

The parameter vector is now given by $\boldsymbol{x} = [\delta\alpha^T \;\; \hat{\boldsymbol{p}}^T]^T$, and the covariance is defined by $P = E\{\boldsymbol{x}\boldsymbol{x}^T\} - E\{\boldsymbol{x}\}E\{\boldsymbol{x}\}^T$. When Eq. (8) is substituted into Eq. (6), and after the appropriate partials are taken, the following optimal error covariance can be derived:

$$P = \begin{bmatrix} -\displaystyle\sum_{i=1}^{N}\sigma_i^{-2}[A\boldsymbol{r}_i\times]^2 & \displaystyle\sum_{i=1}^{N}\sigma_i^{-2}\zeta_i A[\boldsymbol{r}_i\times] \\ \displaystyle\sum_{i=1}^{N}\sigma_i^{-2}\zeta_i[\boldsymbol{r}_i\times]^T A^T & -\displaystyle\sum_{i=1}^{N}\sigma_i^{-2}\zeta_i^2[\boldsymbol{r}_i\times]^2 \end{bmatrix}^{-1} \equiv F^{-1} \tag{10}$$

with obvious definition for $F$, and where

$$\zeta_i \equiv \left[(X_i - X_c)^2 + (Y_i - Y_c)^2 + (Z_i - Z_c)^2\right]^{-\frac{1}{2}} \tag{11}$$

The terms $A$ and $\boldsymbol{r}_i$ are evaluated at their respective true values (although in practice the estimates are used). Note that Eq. (10) gives the Cramér–Rao lower bound (see Ref. 9). [Any estimator whose error covariance is equivalent to Eq. (10) is an efficient, that is, optimal, estimator.] Also, Eq. (10) is directly used in the GLSDC process and predictive filter solution.[3]

The matrix $F$ in Eq. (10) must have rank six for $P$ to exist. The remainder of this paper is devoted to the analysis of the matrix $F$ for a number of vector observation cases. We first prove that the rank of $F$ is independent of the attitude matrix $A$. Because $A$ is a proper orthogonal matrix, then $AA^T = A^TA = I_{3\times 3}$. Also, the following identity is helpful:

$$[A\boldsymbol{r}\times] = A[\boldsymbol{r}\times]A^T \tag{12}$$

Next, a similarity transformation is performed using the following orthogonal matrix:

$$\mathcal{M} = \begin{bmatrix} A & 0_{3\times 3} \\ 0_{3\times 3} & I_{3\times 3} \end{bmatrix} \tag{13}$$

where $0_{3\times3}$ is a $3\times3$ zero matrix. Defining $\mathcal{F}\equiv\mathcal{M}^T F\mathcal{M}$ and using the identity in Eq. (12) gives

$$\mathcal{F}=\begin{bmatrix} -\sum_{i=1}^N \sigma_i^{-2}[\boldsymbol{r}_i\times]^2 & \sum_{i=1}^N \sigma_i^{-2}\zeta_i[\boldsymbol{r}_i\times] \\ \sum_{i=1}^N \sigma_i^{-2}\zeta_i[\boldsymbol{r}_i\times]^T & -\sum_{i=1}^N \sigma_i^{-2}\zeta_i^2[\boldsymbol{r}_i\times]^2 \end{bmatrix} \tag{14}$$

Because rank $(\mathcal{M})=6$, then rank $(F)=\mathrm{rank}(\mathcal{F})$, which indicates that the degree of observability, that is, the rank of $F$, of the system is independent of the attitude matrix. This intuitively makes sense because the orientation of the body with respect to the beacon LOS sources does not affect the overall observability. (It does, however, affect the relative degree of observability of each axis component.)

## One-Vector Observation Case

In this section the one-vector case is analyzed. Although in practice one observation would not be used, this case is worthy of study because as the range to multiple beacons becomes large, the angular separation decreases and the beacons ultimately approach collocation. The result is a geometric dilution of precision, and ultimately, a loss of observability analogous to the one-beacon case. The rank of the information matrix is first investigated. For this case $\mathcal{F}$ is given by

$$\mathcal{F}=\sigma^{-2}\begin{bmatrix} -[\boldsymbol{r}\times]^2 & \zeta[\boldsymbol{r}\times] \\ \zeta[\boldsymbol{r}\times]^T & -\zeta^2[\boldsymbol{r}\times]^2 \end{bmatrix}\equiv\sigma^{-2}M \tag{15}$$

with obvious definition for $M$. Using the matrix

$$\mathcal{N}=\begin{bmatrix} \zeta[\boldsymbol{r}\times]^T & I_{3\times3} \\ I_{3\times3} & 0_{3\times3} \end{bmatrix} \tag{16}$$

we have

$$\mathcal{N}^T M\mathcal{N}=\begin{bmatrix} 0_{3\times3} & 0_{3\times3} \\ 0_{3\times3} & -[\boldsymbol{r}\times]^2 \end{bmatrix} \tag{17}$$

where the following identities were used for any unit vector $\boldsymbol{r}$:

$$[\boldsymbol{r}\times]^3=-[\boldsymbol{r}\times] \tag{18a}$$

$$[\boldsymbol{r}\times]^4=-[\boldsymbol{r}\times]^2 \tag{18b}$$

Therefore, because $\mathrm{rank}(\mathcal{N})=6$, then $\mathrm{rank}(M)$, and ultimately the rank of $F$, is given by the $\mathrm{rank}(-[\boldsymbol{r}\times]^2)$. The matrix $-[\boldsymbol{r}\times]^2=I_{3\times3}-\boldsymbol{r}\boldsymbol{r}^T$ is the projection matrix onto the space perpendicular to $\boldsymbol{r}$ and has rank 2. This indicates that only two pieces of information are given using one-vector observation.

The eigenvalues of $M$ are given by solving the following equation:

$$\det(\lambda I_{3\times3}-M)=\det\begin{bmatrix} \lambda I_{3\times3}+[\boldsymbol{r}\times]^2 & -\zeta[\boldsymbol{r}\times] \\ -\zeta[\boldsymbol{r}\times]^T & \lambda I_{3\times3}+\zeta^2[\boldsymbol{r}\times]^2 \end{bmatrix}=0 \tag{19}$$

Performing the matrix determinant operation gives

$$\begin{aligned}\det(\lambda I_{3\times3}-M) &= \det\left\{\left(\lambda I_{3\times3}+[\boldsymbol{r}\times]^2\right)\left(\lambda I_{3\times3}+\zeta^2[\boldsymbol{r}\times]^2\right)+\zeta^2[\boldsymbol{r}\times]^2\right\} \\ &= \det\left\{\lambda^2 I_{3\times3}+\lambda(1+\zeta^2)[\boldsymbol{r}\times]^2+\zeta^2[\boldsymbol{r}\times]^4+\zeta^2[\boldsymbol{r}\times]^2\right\}\end{aligned} \tag{20}$$

Next, using the identity in Eq. (18b) yields

$$\det(\lambda I_{3\times3}-M)=\lambda^3\det\left\{\lambda I_{3\times3}+(1+\zeta^2)[\boldsymbol{r}\times]^2\right\} \tag{21}$$

Clearly, three eigenvalues of $M$ are zero. The eigenvalues of $-(1+\zeta^2)[\boldsymbol{r}\times]^2$ are well known, and are given by 0 and twice repeated $(1+\zeta^2)$. Therefore, the eigenvalues of $\mathcal{F}=\sigma^{-2}M$ are given by

$$\lambda_1=\lambda_2=\lambda_3=\lambda_4=0,\qquad \lambda_5=\lambda_6=\sigma^{-2}(1+\zeta^2) \tag{22}$$

Because the eigenvalues of a matrix are unaffected by a similarity transformation, Eq. (22) also gives the eigenvalues of $F$.

To calculate the eigenvectors of $F$, we first state a well-known property of a symmetric matrix. Let $\Upsilon$ be an $n\times n$ symmetric matrix. There exists an orthogonal matrix $Z$ such that $Z^T\Upsilon Z=D$, where $D$ is a diagonal matrix with the characteristic roots of $\Upsilon$. This also states that a symmetric matrix is similar to a diagonal matrix.[10] Note that if some eigenvalue has multiplefold degeneracy (as in the present case), one can find an orthogonal basis in the subspace spanned by its eigenvectors. Therefore, $\mathcal{F}=W\mathrm{diag}[\lambda_1\ \lambda_2\ \lambda_3\ \lambda_4\ \lambda_5\ \lambda_6]W^T$, where $W=[\boldsymbol{w}_1\ \boldsymbol{w}_2\ \boldsymbol{w}_3\ \boldsymbol{w}_4\ \boldsymbol{w}_5\ \boldsymbol{w}_6]$ is an orthogonal matrix, and $\boldsymbol{w}_i$, $i=1,2,\ldots,6$, are $6\times1$ orthogonal eigenvectors. We now calculate the eigenvectors $\boldsymbol{w}_5$ and $\boldsymbol{w}_6$, which correspond to the eigenvalues $\lambda_5$ and $\lambda_6$ in Eq. (22), respectively. For the eigenvalue $\lambda_5$, we have

$$\mathcal{F}\boldsymbol{w}_5=\lambda_5\boldsymbol{w}_5=\sigma^{-2}(1+\zeta^2)\boldsymbol{w}_5 \tag{23}$$

From Eq. (15), $\mathcal{F}=\sigma^{-2}M$; hence, $M\boldsymbol{w}_5=(1+\zeta^2)\boldsymbol{w}_5$. Let $\boldsymbol{w}_5$ and $\boldsymbol{w}_6$ be partitioned into

$$\boldsymbol{w}_5\equiv\begin{bmatrix}\boldsymbol{w}_{51}\\ \boldsymbol{w}_{52}\end{bmatrix},\qquad \boldsymbol{w}_6\equiv\begin{bmatrix}\boldsymbol{w}_{61}\\ \boldsymbol{w}_{62}\end{bmatrix} \tag{24}$$

where $\boldsymbol{w}_{51}$, $\boldsymbol{w}_{52}$, $\boldsymbol{w}_{61}$ and $\boldsymbol{w}_{62}$ are $3\times1$ partition vectors of $\boldsymbol{w}_5$ and $\boldsymbol{w}_6$, respectively. From the definition of $M$ in Eq. (15) and using the partitioned eigenvector in Eq. (24), the following two equations are given:

$$-[\boldsymbol{r}\times]^2\boldsymbol{w}_{51}+\zeta[\boldsymbol{r}\times]\boldsymbol{w}_{52}=(1+\zeta^2)\boldsymbol{w}_{51} \tag{25a}$$

$$-\zeta[\boldsymbol{r}\times]\boldsymbol{w}_{51}-\zeta^2[\boldsymbol{r}\times]^2\boldsymbol{w}_{52}=(1+\zeta^2)\boldsymbol{w}_{52} \tag{25b}$$

Simultaneously solving Eqs. (25a) and (25b) gives

$$\boldsymbol{w}_{51}\perp\boldsymbol{r} \tag{26a}$$

$$\boldsymbol{w}_{52}=-\zeta[\boldsymbol{r}\times]\boldsymbol{w}_{51} \tag{26b}$$

This states that both $\boldsymbol{w}_{51}$ and $\boldsymbol{w}_{52}$ lie in the plane perpendicular to $\boldsymbol{r}$. Also, clearly, $\boldsymbol{w}_{51}\perp\boldsymbol{w}_{52}$, which means that the vectors $\boldsymbol{w}_{51}$, $\boldsymbol{w}_{52}$, and $\boldsymbol{r}$ form an orthogonal set.

To determine the eigenvector $\boldsymbol{w}_5$, the vectors $\boldsymbol{r}$ and $\boldsymbol{w}_{51}$ are first given in component form by $\boldsymbol{r}=[r_1\ r_2\ r_3]^T$ and $\boldsymbol{w}_{51}=[w_{51}\ w_{52}\ w_{53}]^T$, respectively. At least one component of $\boldsymbol{r}$ must be nonzero. We assume that $r_1\neq0$, but the argument goes through with only minor modification for any nonzero component. Because $\boldsymbol{w}_{51}^T\boldsymbol{r}=0$, and assuming $r_1\neq0$, then $w_{51}=-(w_{52}r_2+w_{53}r_3)/r_1$. Next, without loss in generality, we can assume that $w_{52}=1$ and $w_{53}=0$, so that $\boldsymbol{w}_{51}=[-r_2/r_1\ \ 1\ \ 0]^T$. Therefore, by the use of Eq. (26b), the normalized vector for $\boldsymbol{w}_5$ is given by

$$\boldsymbol{w}_5=\boldsymbol{a}/\|\boldsymbol{a}\| \tag{27}$$

where

$$\boldsymbol{a}\equiv\left[-r_2/r_1\quad 1\quad 0\quad \zeta r_3\quad \zeta r_2 r_3/r_1\quad -\zeta\left(r_2^2/r_1+r_1\right)\right]^T \tag{28}$$

In a similar fashion, by the use of $\boldsymbol{w}_5^T\boldsymbol{w}_6=0$, the normalized vector for $\boldsymbol{w}_6$ is given by

$$\boldsymbol{w}_6=\boldsymbol{b}/\|\boldsymbol{b}\| \tag{29}$$

where

$$\boldsymbol{b}\equiv\left[r_1^2r_3\quad r_1r_2r_3\quad -r_1\left(r_1^2+r_2^2\right)\quad \zeta r_1r_2\quad -\zeta r_1^2\quad 0\right]^T \tag{30}$$

If $r_1=0$, then other eigenvectors can be found by using the nonzero component values of $\boldsymbol{r}$.

The next step involves determining the eigenvectors of $F$, which is decomposed as $F=V\mathrm{diag}[\lambda_1\ \lambda_2\ \lambda_3\ \lambda_4\ \lambda_5\ \lambda_6]V^T$, where $V=[\boldsymbol{v}_1\ \boldsymbol{v}_2\ \boldsymbol{v}_3\ \boldsymbol{v}_4\ \boldsymbol{v}_5\ \boldsymbol{v}_6]$ is an orthogonal matrix and $\boldsymbol{v}_i$, $i=1,2,\ldots,6$, are $6\times1$ orthogonal eigenvectors. By the use of $F=\mathcal{M}\mathcal{F}\mathcal{M}^T$, where $\mathcal{M}$ is defined by Eq. (13), the eigenvectors

$\boldsymbol{v}_5$ and $\boldsymbol{v}_6$ associated with the eigenvalues $\lambda_5$ and $\lambda_6$, respectively, are given by

$$\boldsymbol{v}_5 \equiv \begin{bmatrix} \boldsymbol{v}_{51} \\ \boldsymbol{v}_{52} \end{bmatrix} = \mathcal{M}\boldsymbol{w}_5 = \begin{bmatrix} A\boldsymbol{w}_{51} \\ \boldsymbol{w}_{52} \end{bmatrix} \tag{31a}$$

$$\boldsymbol{v}_6 \equiv \begin{bmatrix} \boldsymbol{v}_{61} \\ \boldsymbol{v}_{62} \end{bmatrix} = \mathcal{M}\boldsymbol{w}_6 = \begin{bmatrix} A\boldsymbol{w}_{61} \\ \boldsymbol{w}_{62} \end{bmatrix} \tag{31b}$$

The vectors $\boldsymbol{v}_5$ and $\boldsymbol{v}_6$ give information of the observable components for attitude and position. Each vector is equally observable because the eigenvalues are repeated. Position and attitude information can not be decoupled because $\|\boldsymbol{r}\| = 1 \neq 0$, which means that with one observation no useful information can be provided. This is in sharp contrast to standard attitude determination results using one-vector observation, in which one-vector observation provides two-axis attitude information.[11] The analysis in this section also indicates that, for the multiple-beacon case, as the angular separation of the beacons decreases (approaching collocation) the physical meaning of the attitude and position results becomes skewed.

## Two-Vector Observation Case

In this section, the two-vector case is analyzed. We assume that the two vectors $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ are noncollinear. Unlike the one-vector case, the two-vector case does provide some physical insights that are useful for beacon location studies. For the two-vector case, the matrix $\mathcal{F}$ in Eq. (14) is given by

$$\mathcal{F} = \sum_{i=1}^{2} \mathcal{F}_i \tag{32}$$

where $\mathcal{F}_i$ is given by

$$\mathcal{F}_i = \sigma_i^{-2} L_i L_i^T \tag{33}$$

with

$$L_i \equiv \begin{bmatrix} -[\boldsymbol{r}_i\times] \\ \zeta_i[\boldsymbol{r}_i\times]^2 \end{bmatrix} \tag{34}$$

Rearranging the partitioned elements of $L_i$ yields

$$\mathcal{F} = \sigma_1^{-2} L_1 L_1^T + \sigma_2^{-2} L_2 L_2^T \equiv L L^T \tag{35}$$

with

$$L \equiv \begin{bmatrix} -\sigma_1^{-1}[\boldsymbol{r}_1\times] & -\sigma_2^{-1}[\boldsymbol{r}_2\times] \\ \sigma_1^{-1}\zeta_1[\boldsymbol{r}_1\times]^2 & \sigma_2^{-1}\zeta_2[\boldsymbol{r}_2\times]^2 \end{bmatrix} \tag{36}$$

where the identities in Eq. (18) were used in the preceding quantities. Clearly, we now have $\text{rank}(\mathcal{F}) = \text{rank}(L)$.

We now discuss the rank of the matrix $L$. Hildebrand[12] shows that the rank of a $q \times n$ matrix $C$ $(q \geq n)$ is $n - m$, where $m$ is the maximum number of orthogonal vectors $\boldsymbol{y}$ that satisfy $C\boldsymbol{y} = \boldsymbol{0}$. For the two-vector case, consider the conditions for $L^T\boldsymbol{y} = \boldsymbol{0}$, with $\boldsymbol{y} \neq \boldsymbol{0}$, to be satisfied. Using the partitioned elements of $L$ yields

$$[\boldsymbol{r}_1\times]\boldsymbol{y}_1 + \zeta_1[\boldsymbol{r}_1\times]^2\boldsymbol{y}_2 = \boldsymbol{0} \tag{37a}$$

$$[\boldsymbol{r}_2\times]\boldsymbol{y}_1 + \zeta_2[\boldsymbol{r}_2\times]^2\boldsymbol{y}_2 = \boldsymbol{0} \tag{37b}$$

where

$$\boldsymbol{y} \equiv \begin{bmatrix} \boldsymbol{y}_1^T & \boldsymbol{y}_2^T \end{bmatrix}^T$$

The general relations for $\boldsymbol{y}_1$ and $\boldsymbol{y}_2$ that satisfy Eq. (37) are given by

$$\boldsymbol{y}_1 = -\zeta_1[\boldsymbol{r}_1\times]\boldsymbol{y}_2 + c_1\boldsymbol{r}_1 \tag{38a}$$

$$\boldsymbol{y}_1 = -\zeta_2[\boldsymbol{r}_2\times]\boldsymbol{y}_2 + c_2\boldsymbol{r}_2 \tag{38b}$$

where $c_1$ and $c_2$ are arbitrary constants. Subtracting Eq. (38a) from Eq. (38b) gives

$$(\zeta_1[\boldsymbol{r}_1\times] - \zeta_2[\boldsymbol{r}_2\times])\boldsymbol{y}_2 = c_1\boldsymbol{r}_1 - c_2\boldsymbol{r}_2 \tag{39}$$

We first consider the case where $c_1\boldsymbol{r}_1 - c_2\boldsymbol{r}_2 = \boldsymbol{0}$. Because it is assumed that $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ are noncollinear, then $c_1 = c_2 = 0$; thus, we have

$$\boldsymbol{y}_2 = \pm(\zeta_1\boldsymbol{r}_1 - \zeta_2\boldsymbol{r}_2) \tag{40}$$

Therefore, the vector $\boldsymbol{y}_2$ is contained in the plane given by $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$. Next, we consider the case where $c_1\boldsymbol{r}_1 - c_2\boldsymbol{r}_2 \neq \boldsymbol{0}$. From Eq. (39), the quantity $c_1\boldsymbol{r}_1 - c_2\boldsymbol{r}_2$ must be perpendicular to both $\boldsymbol{y}_2$ and to $(\zeta_1\boldsymbol{r}_1 - \zeta_2\boldsymbol{r}_2)$. Therefore, another solution for $\boldsymbol{y}_2$, denoted by $\boldsymbol{y}_2'$, is given by

$$\boldsymbol{y}_2' = -(\zeta_1[\boldsymbol{r}_1\times] - \zeta_2[\boldsymbol{r}_2\times])(c_1\boldsymbol{r}_1 - c_2\boldsymbol{r}_2) \tag{41}$$

Note that $\boldsymbol{y}_2$ and $\boldsymbol{y}_2'$ are orthogonal vectors. Equation (38) can be used to find $\boldsymbol{y}_1$ and $\boldsymbol{y}_1'$. Also, $\boldsymbol{y}$ and $\boldsymbol{y}'$, where

$$\boldsymbol{y}' \equiv \begin{bmatrix} \boldsymbol{y}_1'^T & \boldsymbol{y}_2'^T \end{bmatrix}^T$$

are orthogonal vectors. Therefore, the maximum number of orthogonal vectors $\boldsymbol{y}$ that satisfy $L^T\boldsymbol{y} = \boldsymbol{0}$ is two. Hence, $\text{rank}(L) = \text{rank}(F) = 6 - 2 = 4$. Therefore, four quantities are observable using two-vector observations.

Reference 3 shows that out of these four observable quantities, one axis of attitude and one axis of position information can be determined. (The remaining two quantities must be a combination of attitude and position.) This states that two out of the four observable eigenvectors of the matrix $F$ can be decoupled in attitude and position. The results are summarized here for completeness. We first partition the information matrix $F$ into $3 \times 3$ submatrices as

$$F = P^{-1} = \begin{bmatrix} F_{11} & F_{12} \\ F_{12}^T & F_{22} \end{bmatrix}, \qquad P = \begin{bmatrix} \mathcal{P}_{11}^{-1} & \mathcal{P}_{12}^{-1} \\ \mathcal{P}_{12}^{-T} & \mathcal{P}_{22}^{-1} \end{bmatrix} \tag{42}$$

with obvious definitions for $F_{11}$, $F_{12}$, and $F_{22}$ from Eq. (10). The relationships between $\mathcal{P}_{11}$, $\mathcal{P}_{12}$, and $\mathcal{P}_{22}$ and $F_{11}$, $F_{12}$, and $F_{22}$ are given by[13]

$$\mathcal{P}_{11} = \left(F_{11} - F_{12}F_{22}^{-1}F_{12}^T\right) \tag{43a}$$

$$\mathcal{P}_{12} = F_{11}^{-1}F_{12}\left(F_{12}^T F_{11}^{-1} F_{12} - F_{22}\right) \tag{43b}$$

$$\mathcal{P}_{22} = \left(F_{22} - F_{12}^T F_{11}^{-1} F_{12}\right) \tag{43c}$$

The matrix $\mathcal{P}_{11}$ corresponds to the attitude information, and the matrix $\mathcal{P}_{22}$ corresponds to the position information. The matrix $\mathcal{P}_{11}$ can be shown to be given by

$$\mathcal{P}_{11} = A\mathcal{G}A^T \tag{44}$$

where

$$\mathcal{G} = \left[1/\left(\tilde{\sigma}_1^2 + \tilde{\sigma}_2^2\right)\right]\boldsymbol{g}\boldsymbol{g}^T \tag{45}$$

with

$$\boldsymbol{g} = \begin{bmatrix} \pm\left\{\left(\rho_2^2 + \rho_3^2\right) - \dfrac{\|\boldsymbol{\rho}\|^2\gamma_1^2}{\|\boldsymbol{\gamma}\|^2}\right\}^{\frac{1}{2}} \\ \pm\left\{\left(\rho_1^2 + \rho_3^2\right) - \dfrac{\|\boldsymbol{\rho}\|^2\gamma_2^2}{\|\boldsymbol{\gamma}\|^2}\right\}^{\frac{1}{2}} \\ \pm\left\{\left(\rho_1^2 + \rho_2^2\right) - \dfrac{\|\boldsymbol{\rho}\|^2\gamma_3^2}{\|\boldsymbol{\gamma}\|^2}\right\}^{\frac{1}{2}} \end{bmatrix} \tag{46}$$

$$\boldsymbol{\rho} = \boldsymbol{\beta}_1 - \boldsymbol{\beta}_2 \tag{47a}$$

$$\boldsymbol{\gamma} = \boldsymbol{\beta}_1 \times \boldsymbol{\beta}_2 \tag{47b}$$

$$\boldsymbol{\beta}_i \equiv \begin{bmatrix} X_i - X_c \\ Y_i - Y_c \\ Z_i - Z_c \end{bmatrix}, \qquad i = 1, 2 \tag{47c}$$

$$\tilde{\sigma}_i^2 \equiv \left[(X_i - X_c)^2 + (Y_i - Y_c)^2 + (Z_i - Z_c)^2\right]\sigma_i^2, \qquad i = 1, 2 \tag{47d}$$

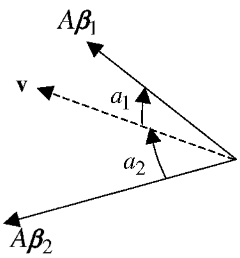


**Fig. 2 Weighted average relation for attitude observability.**

An eigenvalue/eigenvector decomposition of Eq. (44) can be used to assess the observability. The eigenvalues of Eq. (44) are given by $(0, 0, [\tilde{\sigma}_1^2 + \tilde{\sigma}_2^2]^{-1}\|\boldsymbol{\rho}\|^2)$, and the eigenvector associated with the nonzero eigenvalue is given by $\boldsymbol{v} = A\boldsymbol{g}/\|\boldsymbol{g}\|$, which defines the axis of rotation for the observable attitude angle. The eigenvector can easily be shown to lie in the plane of the two body vector observations because $\boldsymbol{v}^T A(\boldsymbol{\beta}_1 \times \boldsymbol{\beta}_2) = 0$. This vector is, in essence, a weighted average of the body observations with

$$\|A\boldsymbol{\beta}_1\| \cos a_1 = \|A\boldsymbol{\beta}_2\| \cos a_2 \tag{48}$$

where $a_1$ is the angle between $A\boldsymbol{\beta}_1$ and $\boldsymbol{v}$, and $a_2$ is the angle between $A\boldsymbol{\beta}_2$ and $\boldsymbol{v}$, as shown in Fig. 2. (Here $a_1 + a_2$ is the angle between $A\boldsymbol{\beta}_1$ and $A\boldsymbol{\beta}_2$.) Equation (48) indicates that the observable axis of rotation is closer to the vector with less length.

In a similar fashion, the position information matrix can be shown to be given by

$$\mathcal{P}_{22} = \left[1\big/\left(\sigma_1^2 + \sigma_2^2\right)\right]\boldsymbol{h}\boldsymbol{h}^T \tag{49}$$

with

$$\boldsymbol{h} = \begin{bmatrix} \pm\left\{\left(\varrho_2^2 + \varrho_3^2\right) - \|\boldsymbol{\varrho}\|^2\vartheta_1^2\big/\|\boldsymbol{\vartheta}\|^2\right\}^{\frac{1}{2}} \\ \pm\left\{\left(\varrho_1^2 + \varrho_3^2\right) - \|\boldsymbol{\varrho}\|^2\vartheta_2^2\big/\|\boldsymbol{\vartheta}\|^2\right\}^{\frac{1}{2}} \\ \pm\left\{\left(\varrho_1^2 + \varrho_2^2\right) - \|\boldsymbol{\varrho}\|^2\vartheta_3^2\big/\|\boldsymbol{\vartheta}\|^2\right\}^{\frac{1}{2}} \end{bmatrix} \tag{50}$$

$$\boldsymbol{\varrho} = \boldsymbol{\delta}_1 - \boldsymbol{\delta}_2 \tag{51a}$$

$$\boldsymbol{\vartheta} = \boldsymbol{\delta}_1 \times \boldsymbol{\delta}_2 \tag{51b}$$

$$\boldsymbol{\delta}_i = \boldsymbol{\beta}_i\big/\|\boldsymbol{\beta}_i\|^2, \qquad i = 1, 2 \tag{51c}$$

The eigenvalues of Eq. (49) are given by $(0, 0, [\sigma_1^2 + \sigma_2^2]^{-1}\|\boldsymbol{\varrho}\|^2)$, and the eigenvector associated with the nonzero eigenvalue is given by $\boldsymbol{w} = \boldsymbol{h}/\|\boldsymbol{h}\|$, which defines the observable position axis. The eigenvector can be shown to lie in the plane of the two reference vectors because $\boldsymbol{w}^T(\boldsymbol{\beta}_1 \times \boldsymbol{\beta}_2) = 0$. The weighted average relationship for the observable position axis is given by

$$\|\boldsymbol{\beta}_1\|/\cos\alpha_1 = \|\boldsymbol{\beta}_2\|/\cos\alpha_2 \tag{52}$$

where $\alpha_1$ is the angle between $\boldsymbol{\beta}_1$ and $\boldsymbol{w}$, and $\alpha_2$ is the angle between $\boldsymbol{\beta}_2$ and $\boldsymbol{w}$ ($\alpha_1 + \alpha_2$ is the angle between $\boldsymbol{\beta}_1$ and $\boldsymbol{\beta}_2$). Equation (52) indicates that the observable position axis is closer to the vector with greater length, which intuitively makes sense because the position solution is more sensitive to the magnitude of the vectors. A slight change in the largest vector produces more change in the position than the same change in the smallest vector. Also, if $\|\boldsymbol{\beta}_1\| = \|\boldsymbol{\beta}_2\|$, or if $\boldsymbol{\beta}_1^T\boldsymbol{\beta}_2 = 0$, then the eigenvector reduces to $\boldsymbol{w} = \pm(\boldsymbol{\beta}_1 + \boldsymbol{\beta}_2)/\|\boldsymbol{\beta}_1 + \boldsymbol{\beta}_2\|$, which is the bisector of the reference vectors. As before, the information given by the two observation vectors is used to calculate the part of the attitude needed to compute the observable position.

Comparing Eq. (48) to Eq. (52) indicates that the beacon that is closest to the target provides the most attitude information, but has the least position information. (This is due to the inverse relationship between them.) The converse is true as well, that is, the beacon that is farthest from the target provides the most position information, but has the least attitude information. (See Ref. 3 for more details.) The covariance analysis can be useful to trade off the relative importance between attitude and position requirements with two-vector observations.

## Three-Vector Observation Case

In this section the three-vector case is analyzed. We assume that any two of the vectors $\boldsymbol{r}_1$, $\boldsymbol{r}_2$, or $\boldsymbol{r}_3$ are noncollinear. We will show that the covariance matrix in this case is full rank for most cases. In the three-vector case, the matrix $\mathcal{F}$ from Eq. (14) is given by

$$\mathcal{F} = LL^T \tag{53}$$

with

$$L \equiv \begin{bmatrix} -\sigma_1^{-1}[\boldsymbol{r}_1\times] & -\sigma_2^{-1}[\boldsymbol{r}_2\times] & -\sigma_3^{-1}[\boldsymbol{r}_3\times] \\ \sigma_1^{-1}\zeta_1[\boldsymbol{r}_1\times]^2 & \sigma_2^{-1}\zeta_2[\boldsymbol{r}_2\times]^2 & \sigma_3^{-1}\zeta_3[\boldsymbol{r}_3\times]^2 \end{bmatrix} \tag{54}$$

As before, the rank of $\mathcal{F}$, and ultimately the rank of $F$, can be determined by considering the conditions for $L^T\boldsymbol{y} = \boldsymbol{0}$, with $\boldsymbol{y} \neq \boldsymbol{0}$, to be satisfied. Using the partitioned elements of $L$ yields

$$[\boldsymbol{r}_1\times]\boldsymbol{y}_1 + \zeta_1[\boldsymbol{r}_1\times]^2\boldsymbol{y}_2 = \boldsymbol{0} \tag{55a}$$

$$[\boldsymbol{r}_2\times]\boldsymbol{y}_1 + \zeta_2[\boldsymbol{r}_2\times]^2\boldsymbol{y}_2 = \boldsymbol{0} \tag{55b}$$

$$[\boldsymbol{r}_3\times]\boldsymbol{y}_1 + \zeta_3[\boldsymbol{r}_3\times]^2\boldsymbol{y}_2 = \boldsymbol{0} \tag{55c}$$

where

$$\boldsymbol{y} \equiv \begin{bmatrix}\boldsymbol{y}_1^T & \boldsymbol{y}_2^T\end{bmatrix}^T$$

The general relations for $\boldsymbol{y}_1$ and $\boldsymbol{y}_2$ that satisfy Eq. (55) are given by

$$\boldsymbol{y}_1 = -\zeta_1[\boldsymbol{r}_1\times]\boldsymbol{y}_2 + c_1\boldsymbol{r}_1 \tag{56a}$$

$$\boldsymbol{y}_1 = -\zeta_2[\boldsymbol{r}_2\times]\boldsymbol{y}_2 + c_2\boldsymbol{r}_2 \tag{56b}$$

$$\boldsymbol{y}_1 = -\zeta_3[\boldsymbol{r}_3\times]\boldsymbol{y}_2 + c_3\boldsymbol{r}_3 \tag{56c}$$

where $c_1$, $c_2$, and $c_3$ are arbitrary constants. Equation (56) can be written in matrix form as

$$D\boldsymbol{y} = \boldsymbol{z} \tag{57}$$

where

$$D \equiv \begin{bmatrix} I_{3\times3} & \zeta_1[\boldsymbol{r}_1\times] \\ I_{3\times3} & \zeta_2[\boldsymbol{r}_2\times] \\ I_{3\times3} & \zeta_3[\boldsymbol{r}_3\times] \end{bmatrix} \tag{58a}$$

$$\boldsymbol{z} = \begin{bmatrix} c_1\boldsymbol{r}_1 \\ c_2\boldsymbol{r}_2 \\ c_3\boldsymbol{r}_3 \end{bmatrix} \tag{58b}$$

A solution to Eq. (57) exists if and only if the rank of the coefficient matrix $D$ is equal to the rank of the augmented matrix $[D\ \ \boldsymbol{z}]$ (Ref. 14). From this theorem, the following scenarios are possible:

1) If $\mathrm{rank}(D) = 6$ and $\mathrm{rank}(D) = \mathrm{rank}([D\ \ \boldsymbol{z}])$, where $\boldsymbol{z} \neq \boldsymbol{0}$, then a solution to Eq. (57) exists, and a nonzero $\boldsymbol{y}$ can be found such that $L^T\boldsymbol{y} = \boldsymbol{0}$. Therefore, $\mathrm{rank}(L) = \mathrm{rank}(F) < 6$.

2) If $\mathrm{rank}(D) = 6$ and $\mathrm{rank}(D) \neq \mathrm{rank}([D\ \ \boldsymbol{z}])$, where $\boldsymbol{z} \neq \boldsymbol{0}$, then a solution to Eq. (57) cannot be determined, and a nonzero $\boldsymbol{y}$ cannot be found such that $L^T\boldsymbol{y} = \boldsymbol{0}$. Therefore, $\mathrm{rank}(L) = \mathrm{rank}(F) = 6$.

3) If $\mathrm{rank}(D) < 6$, then certainly a nonzero $\boldsymbol{y}$ exists such that Eq. (57) is satisfied, and $\mathrm{rank}(L) = \mathrm{rank}(F) < 6$.

We now discuss the properties of the matrix $[D\ \ \boldsymbol{z}]$. Through elementary row operations, this matrix can be shown to be similar to

$$[D\ \ \boldsymbol{z}] \sim \begin{bmatrix} I_{3\times3} & \zeta_1[\boldsymbol{r}_1\times] & c_1\boldsymbol{r}_1 \\ 0_{3\times3} & [\boldsymbol{u}_1\times] & \boldsymbol{\eta}_1 \\ 0_{3\times3} & [\boldsymbol{u}_2\times] & \boldsymbol{\eta}_2 \end{bmatrix} \tag{59}$$

where

$$\boldsymbol{u}_1 \equiv \zeta_2 \boldsymbol{r}_2 - \zeta_1 \boldsymbol{r}_1 \tag{60a}$$

$$\boldsymbol{u}_2 \equiv \zeta_3 \boldsymbol{r}_3 - \zeta_1 \boldsymbol{r}_1 \tag{60b}$$

$$\boldsymbol{\eta}_1 \equiv c_2 \boldsymbol{r}_2 - c_1 \boldsymbol{r}_1 \tag{60c}$$

$$\boldsymbol{\eta}_2 \equiv c_3 \boldsymbol{r}_3 - c_1 \boldsymbol{r}_1 \tag{60d}$$

Define the lower partition of matrix in Eq. (59) by

$$Q \equiv \begin{bmatrix} [\boldsymbol{u}_1 \times] & \boldsymbol{\eta}_1 \\ [\boldsymbol{u}_2 \times] & \boldsymbol{\eta}_2 \end{bmatrix} \tag{61}$$

Also, let the following vectors be given in their components as $\boldsymbol{u}_1 = [u_{11} \;\; u_{12} \;\; u_{13}]^T$, $\boldsymbol{u}_2 = [u_{21} \;\; u_{22} \;\; u_{23}]^T$, $\boldsymbol{\eta}_1 = [\eta_{11} \;\; \eta_{12} \;\; \eta_{13}]^T$, and $\boldsymbol{\eta}_2 = [\eta_{21} \;\; \eta_{22} \;\; \eta_{23}]^T$. If $u_{13} \neq 0$, $u_{23} \neq 0$, and $u_{11}u_{23} - u_{13}u_{21} \neq 0$ (if these conditions are not true then other nonzero elements can be used, as discussed later), the matrix $Q$ can be shown to be similar to

$$Q \sim \begin{bmatrix} V & \boldsymbol{\varpi}_1 \\ 0_{3\times 3} & \boldsymbol{\varpi}_2 \end{bmatrix} \tag{62}$$

where

$$V \equiv \begin{bmatrix} 0 & -u_{13} & u_{12} \\ u_{13} & 0 & -u_{11} \\ 0 & 0 & -u_{21} + \dfrac{u_{11}u_{23}}{u_{13}} \end{bmatrix} \tag{63a}$$

$$\boldsymbol{\varpi}_1 \equiv \begin{bmatrix} \eta_{11} \\ \eta_{12} \\ \eta_{22} - \dfrac{u_{23}\eta_{12}}{u_{13}} \end{bmatrix} \tag{63b}$$

$$\boldsymbol{\varpi}_2 \equiv \begin{bmatrix} \eta_{21} - \dfrac{u_{23}\eta_{11}}{u_{13}} + \dfrac{u_{12}u_{23} - u_{13}u_{22}}{u_{11}u_{23} - u_{13}u_{21}}\left(\eta_{22} - \dfrac{u_{23}\eta_{12}}{u_{13}}\right) \\ \eta_{13} + \dfrac{u_{12}\eta_{12}}{u_{13}} + \dfrac{u_{11}\eta_{11}}{u_{13}} \\ \eta_{23} + \dfrac{u_{22}\eta_{22}}{u_{23}} + \dfrac{u_{21}\eta_{21}}{u_{23}} \end{bmatrix} \tag{63c}$$

If $u_{13} \neq 0$ and $u_{11}u_{23} - u_{13}u_{21} \neq 0$, then $\text{rank}(V) = 3$. If $\boldsymbol{\varpi}_2 = \boldsymbol{0}$, then $\text{rank}(D) = \text{rank}([D \;\; z])$. Therefore, if a set of nonzero $c_1$, $c_2$, and $c_3$ can be found such that $\boldsymbol{\varpi}_2 = \boldsymbol{0}$, then a nonzero $\boldsymbol{y}$ can be found such that $L^T\boldsymbol{y} = \boldsymbol{0}$ is true, and so $\text{rank}(L) = \text{rank}(F) < 6$, which states that full observability in attitude and position is not possible. Also, if a set of nonzero $c_1$, $c_2$, and $c_3$ cannot be found such that $\boldsymbol{\varpi}_2 = \boldsymbol{0}$, then a nonzero $\boldsymbol{y}$ cannot be found such that $L^T\boldsymbol{y} = \boldsymbol{0}$ is true, and so $\text{rank}(L) = \text{rank}(F) = 6$, which states that full observability in attitude and position is possible.

The condition $\boldsymbol{\varpi}_2 = \boldsymbol{0}$ can be restated as

$$E\boldsymbol{c} = \boldsymbol{0} \tag{64}$$

where $\boldsymbol{c} = [c_1 \;\; c_2 \;\; c_3]^T$ and

$$E = \begin{bmatrix} -\boldsymbol{r}_1^T\boldsymbol{u}_1 & \boldsymbol{r}_2^T\boldsymbol{u}_1 & 0 \\ -\boldsymbol{r}_1^T\boldsymbol{u}_2 & 0 & \boldsymbol{r}_3^T\boldsymbol{u}_2 \\ e_{31} & e_{32} & e_{33} \end{bmatrix} \tag{65}$$

The quantities $e_{31}$, $e_{32}$, and $e_{33}$ are given by

$$e_{31} = (u_{23} - u_{13})[(u_{11}u_{23} - u_{13}u_{21})r_{11} + (u_{12}u_{23} - u_{13}u_{22})r_{12}] \tag{66a}$$

$$e_{32} = -u_{23}[(u_{11}u_{23} - u_{13}u_{21})r_{21} + (u_{12}u_{23} - u_{13}u_{22})r_{22}] \tag{66b}$$

$$e_{33} = u_{13}[(u_{11}u_{23} - u_{13}u_{21})r_{31} + (u_{12}u_{23} - u_{13}u_{22})r_{32}] \tag{66c}$$

where $\boldsymbol{r}_1 = [r_{11} \;\; r_{12} \;\; r_{13}]^T$, $\boldsymbol{r}_2 = [r_{21} \;\; r_{22} \;\; r_{23}]^T$, and $\boldsymbol{r}_3 = [r_{31} \;\; r_{32} \;\; r_{33}]^T$. If $\boldsymbol{r}_2^T\boldsymbol{u}_1 \neq 0$ and $\boldsymbol{r}_3^T\boldsymbol{u}_2 \neq 0$, the following similarity condition can be obtained through elementary row operations:

$$E \sim \begin{bmatrix} -\boldsymbol{r}_1^T\boldsymbol{u}_1 & \boldsymbol{r}_2^T\boldsymbol{u}_1 & 0 \\ -\boldsymbol{r}_1^T\boldsymbol{u}_2 & 0 & \boldsymbol{r}_3^T\boldsymbol{u}_2 \\ \chi & 0 & 0 \end{bmatrix} \tag{67}$$

where

$$\chi = e_{31} + \left(\boldsymbol{r}_1^T\boldsymbol{u}_1 / \boldsymbol{r}_2^T\boldsymbol{u}_1\right)e_{32} + \left(\boldsymbol{r}_1^T\boldsymbol{u}_2 / \boldsymbol{r}_3^T\boldsymbol{u}_2\right)e_{33} \tag{68}$$

If $\chi \neq 0$, then $\text{rank}(E) = 3$, and Eq. (64) can only be satisfied when $\boldsymbol{c} = \boldsymbol{0}$. Hence, $\text{rank}(L) = \text{rank}(F) = 6$, which gives an observable system. After some algebraic manipulations, $\chi$ can also be shown to be given by

$$\chi = [0 \quad 0 \quad 1][(\boldsymbol{u}_1 \times \boldsymbol{u}_2) \times \boldsymbol{v}] \tag{69}$$

where

$$\boldsymbol{v} \equiv (u_{23} - u_{13})\boldsymbol{r}_1 - \left(\boldsymbol{r}_1^T\boldsymbol{u}_1 / \boldsymbol{r}_2^T\boldsymbol{u}_1\right)u_{23}\boldsymbol{r}_2 + \left(\boldsymbol{r}_1^T\boldsymbol{u}_2 / \boldsymbol{r}_3^T\boldsymbol{u}_2\right)u_{13}\boldsymbol{r}_3 \tag{70}$$

Therefore, $\chi = 0$ when the third component of $[(\boldsymbol{u}_1 \times \boldsymbol{u}_2) \times \boldsymbol{v}]$ is zero, which occurs when $\boldsymbol{u}_1$, $\boldsymbol{u}_2$, and $\boldsymbol{v}$ lie in the same plane that is perpendicular to the object space plane given by $Z = 0$ (Fig. 1). Hence, $\text{rank}(E) < 3$, and a nonzero $\boldsymbol{c}$ can be found that satisfies Eq. (64), which means that the system is not observable because $\text{rank}(L) = \text{rank}(F) < 6$. From the matrix $E$ in Eq. (65), the following cases can easily be proved:

1) If $\boldsymbol{r}_3^T\boldsymbol{u}_2 = 0$ and $e_{33} = 0$, then $\text{rank}(F) < 6$.
2) If $\boldsymbol{r}_2^T\boldsymbol{u}_1 = 0$ and $e_{32} = 0$, then $\text{rank}(F) < 6$.
3) If $\boldsymbol{r}_1^T\boldsymbol{u}_1 = \boldsymbol{r}_2^T\boldsymbol{u}_1 = 0$, then $\text{rank}(F) < 6$.
4) If $\boldsymbol{r}_1^T\boldsymbol{u}_2 = \boldsymbol{r}_3^T\boldsymbol{u}_2 = 0$, then $\text{rank}(F) < 6$.
5) If $\boldsymbol{r}_2^T\boldsymbol{u}_1 = 0$ and $e_{32} \neq 0$, with $\boldsymbol{r}_3^T\boldsymbol{u}_2 \neq 0$ and $\boldsymbol{r}_1^T\boldsymbol{u}_1 \neq 0$, then $\text{rank}(F) = 6$.
6) If $\boldsymbol{r}_3^T\boldsymbol{u}_2 = 0$ and $e_{33} \neq 0$, with $\boldsymbol{r}_2^T\boldsymbol{u}_1 \neq 0$ and $\boldsymbol{r}_1^T\boldsymbol{u}_2 \neq 0$, then $\text{rank}(F) = 6$.

The first two conditions are physically interesting cases because they can be satisfied even if the vectors $\boldsymbol{r}_1$, $\boldsymbol{r}_2$, and $\boldsymbol{r}_3$ are not coplanar, for example, consider the vectors $\boldsymbol{r}_1 = [1 \;\; -2 \;\; 1]^T/\sqrt{6}$, $\boldsymbol{r}_2 = [2 \;\; -2 \;\; 1]^T/3$, and $\boldsymbol{r}_3 = [3 \;\; -1 \;\; 1]^T/\sqrt{11}$, which gives a rank deficient $F$. Also, the third and fourth cases occur only when the three vectors $\zeta_1\boldsymbol{r}_1$, $\zeta_2\boldsymbol{r}_2$, and $\zeta_3\boldsymbol{r}_3$ are parallel to each other with equal magnitude, which violates the assumption made in this section. An obvious rank deficient condition for $Q$ in Eq. (61) exists when $\boldsymbol{u}_1 \times \boldsymbol{u}_2 = \boldsymbol{0}$, which occurs when $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ are parallel.

In the preceding derivations, it has been assumed that $u_{13} \neq 0$, $u_{23} \neq 0$, and $u_{11}u_{23} - u_{13}u_{21} \neq 0$. If these conditions are not true, then the other nonzero elements of $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ can be used to derive similar conditions for unobservability. This yields a condition of unobservability that occurs when the endpoints of the position vectors ($\zeta_1\boldsymbol{r}_1$, $\zeta_2\boldsymbol{r}_2$, and $\zeta_3\boldsymbol{r}_3$) can be connected by a straight line, for example, consider the vectors $\boldsymbol{r}_1 = [1 \;\; 2 \;\; 1]^T/\sqrt{6}$, $\boldsymbol{r}_2 = [1 \;\; 2 \;\; 2]^T/3$, and $\boldsymbol{r}_3 = [1 \;\; 2 \;\; 3]^T/\sqrt{14}$, which gives a rank deficient $F$. Also note that even though $F$ can be shown to have full rank using three-vector observations under most conditions, a unique attitude and position cannot be determined due to a sign ambiguity in the solution. This is difficult to prove analytically, but can be shown by simulation. This scenario is similar to attitude determination results using angle observations.[15]

## More Than Three Observations

In the four-vector case the matrix $\mathcal{F}$ from Eq. (14) is given by

$$\mathcal{F} = LL^T \tag{71}$$

with

$$L \equiv \begin{bmatrix} -\sigma_1^{-1}[\boldsymbol{r}_1\times] & -\sigma_2^{-1}[\boldsymbol{r}_2\times] & -\sigma_3^{-1}[\boldsymbol{r}_3\times] & -\sigma_4^{-1}[\boldsymbol{r}_4\times] \\ \sigma_1^{-1}\zeta_1[\boldsymbol{r}_1\times]^2 & \sigma_2^{-1}\zeta_2[\boldsymbol{r}_2\times]^2 & \sigma_3^{-1}\zeta_3[\boldsymbol{r}_3\times]^2 & \sigma_4^{-1}\zeta_4[\boldsymbol{r}_4\times]^2 \end{bmatrix} \tag{72}$$

As before, the rank of $\mathcal{F}$, and ultimately the rank of $F$, can be determined by considering the conditions for $L^T\boldsymbol{y} = \boldsymbol{0}$, with $\boldsymbol{y} \neq \boldsymbol{0}$, to be satisfied. In a similar fashion as the three-vector case, the conditions for $L^T \boldsymbol{y} = \boldsymbol{0}$ can be written as

$$D\boldsymbol{y} = \boldsymbol{z} \tag{73}$$

where

$$D \equiv \begin{bmatrix} I_{3\times 3} & \zeta_1[\boldsymbol{r}_1 \times] \\ I_{3\times 3} & \zeta_2[\boldsymbol{r}_2 \times] \\ I_{3\times 3} & \zeta_3[\boldsymbol{r}_3 \times] \\ I_{3\times 3} & \zeta_4[\boldsymbol{r}_4 \times] \end{bmatrix} \tag{74a}$$

$$\boldsymbol{z} = \begin{bmatrix} c_1\boldsymbol{r}_1 \\ c_2\boldsymbol{r}_2 \\ c_3\boldsymbol{r}_3 \\ c_4\boldsymbol{r}_4 \end{bmatrix} \tag{74b}$$

A condition for an unobservable system can be derived using the same procedure as in the three-vector case. Similar to the three-vector case, the four-vector case is unobservable when the endpoints of the position vectors can be connected by a straight line. These results are also valid when more than four LOS vectors are used. Furthermore, a unique solution for the attitude and position exists when four beacons are present and the system is observable.[3]

## Trace and Eigenvalues of the Covariance Matrix

In this section, the trace of the covariance matrix, given in Eq. (10), is analyzed. The trace of this matrix is useful to quantify the overall performance of the solution for the attitude and position, that is, a lower trace provides a more overall accurate solution. The matrix $\mathcal{F}$ in Eq. (14) can be written as

$$\mathcal{F} = \sum_{i=1}^{N} \mathcal{F}_i \tag{75}$$

where

$$\mathcal{F}_i = \begin{bmatrix} -\sigma_i^{-2}[\boldsymbol{r}_i \times]^2 & \sigma_i^{-2}\zeta_i[\boldsymbol{r}_i \times] \\ \sigma_i^{-2}\zeta_i[\boldsymbol{r}_i \times]^T & -\sigma_i^{-2}\zeta_i^2[\boldsymbol{r}_i \times]^2 \end{bmatrix} \tag{76}$$

The eigenvalues of $\mathcal{F}_i$ are given by

$$\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0, \qquad \lambda_5 = \lambda_6 = \sigma_i^{-2}\left(1 + \zeta_i^2\right) \tag{77}$$

Then the trace of the information matrix $F$ is given by

$$\operatorname{tr}(F) = \operatorname{tr}(\mathcal{F}) = 2\sum_{i=1}^{N} \sigma_i^{-2}\left(1 + \zeta_i^2\right) \tag{78}$$

where the invariance of the trace through a similarity transformation is used.

We now discuss the properties of the matrix $P$. First a useful theorem is shown. Given two real $n \times n$ symmetric matrices, $A$ and $B$, with $A$ positive definite and $B$ positive semidefinite, there exists a nonsingular $T$ such that $A = TT^T$ and $B = T\Upsilon T^T$, where $\Upsilon$ is a diagonal matrix with elements given by $\Upsilon = \operatorname{diag}[\mu_1 \;\; \mu_2 \;\; \cdots \;\; \mu_n]$ (Ref. 16). The matrix $T$ can be derived using the following procedure. Because $A$ is symmetric and positive definite, a singular value decomposition can be performed so that $A = UQ^2U^T$, where $U$ is an orthogonal matrix and $Q$ is a diagonal matrix of the square roots of the eigenvalues of $A$. Then, compute $C = Q^{-1}U^T BUQ^{-1}$. Because $C$ is symmetric, a singular value decomposition can be performed so that $C = V\Upsilon V^T$. Then $T = UQV$. Let

$$\bar{\mathcal{F}}_3 \equiv \sum_{i=1}^{3} \mathcal{F}_i$$

have full rank, and define

$$\bar{\mathcal{F}}_4 \equiv \sum_{i=1}^{4} \mathcal{F}_i = \bar{\mathcal{F}}_3 + \mathcal{F}_4$$

which also has full rank. Also, $\bar{\mathcal{F}}_3$ and $\bar{\mathcal{F}}_4$ are positive-definite matrices, and $\mathcal{F}_4$ is positive semidefinite. This theorem can be shown to prove easily that if $\mathcal{F}$ has full rank for three-vector observations, then $\mathcal{F}$ has full rank for more than three observations. Now let $\bar{\mathcal{F}}_3 = TT^T = UQ^2U^T$ and let $C = Q^{-1}U^T\mathcal{F}_4UQ^{-1} = V\Upsilon V^T$. Thus, $\mathcal{F}_4 = T\Upsilon T^T$. Therefore, $\bar{\mathcal{F}}_4$ is given by

$$\begin{aligned} \bar{\mathcal{F}}_4 &= \bar{\mathcal{F}}_3 + \mathcal{F}_4 = T\,T^T + T\Upsilon T^T \\ &= T\operatorname{diag}[(1+\mu_1)\;(1+\mu_2)\;\cdots\;(1+\mu_6)]T^T \end{aligned} \tag{79}$$

After some algebraic manipulations, $\bar{\mathcal{F}}_4^{-1}$ can be shown to be given by

$$\bar{\mathcal{F}}_4^{-1} = \bar{\mathcal{F}}_3^{-1} - \Delta\bar{\mathcal{F}} \tag{80}$$

where $\Delta\bar{\mathcal{F}}$ is a positive-semidefinite matrix given by

$$\Delta\bar{\mathcal{F}} = UQ^{-1}V\operatorname{diag}\left[\frac{\mu_1}{1+\mu_1}\,\frac{\mu_2}{1+\mu_2}\cdots\frac{\mu_6}{1+\mu_6}\right]V^TQ^{-1}U^T \tag{81}$$

Using that the trace of the sum of two matrices is given by sum of the trace of each matrix individually, we have $\operatorname{tr}(\bar{\mathcal{F}}_4^{-1}) = \operatorname{tr}(\bar{\mathcal{F}}_3^{-1}) - \operatorname{tr}(\Delta\bar{\mathcal{F}})$. Therefore, because $\operatorname{tr}(\Delta\bar{\mathcal{F}}) > 0$, then $\operatorname{tr}(\bar{\mathcal{F}}_4^{-1}) < \operatorname{tr}(\bar{\mathcal{F}}_3^{-1})$. Because the trace is invariant under a similarity transformation, then the trace of the covariance matrix $P$ in Eq. (10) with four-vector observations is always less than the trace of the covariance using three-vector observations, which intuitively makes sense. This result can be further expanded to multiple observations, that is, the trace of the covariance using $N$ observations is always less than the trace using any number of observations less than $N$.

We now discuss the properties of the eigenvalues of $P$. Consider the following decomposition: $\bar{\mathcal{F}}_3\boldsymbol{x}_i = \lambda_i\boldsymbol{x}_i$ and $\bar{\mathcal{F}}_4\boldsymbol{y}_i = \alpha_i\boldsymbol{y}_i$, $i = 1, 2, \ldots, 6$, where $\lambda_i$ is an eigenvalue of the matrix $\bar{\mathcal{F}}_3$, $\boldsymbol{x}_i$ is the eigenvector of the matrix $\bar{\mathcal{F}}_3$ corresponding to $\lambda_i$, $\alpha_i$ is an eigenvalue of the matrix $\bar{\mathcal{F}}_4$, and $\boldsymbol{y}_i$ is the eigenvector of the matrix $\bar{\mathcal{F}}_4$ corresponding with $\alpha_i$. Because the eigenvectors of a symmetric matrix are orthogonal, $\bar{\mathcal{F}}_3\boldsymbol{y}_i$ is related by

$$\bar{\mathcal{F}}_3\boldsymbol{y}_i = \sum_{j=1}^{6} k_{ij}\lambda_j\boldsymbol{x}_j \tag{82}$$

where the $k_{ij}$ are constants with

$$\sum_{j=1}^{6} k_{ij}^2 = 1$$

Also, $\mathcal{F}_4\boldsymbol{y}_i$ is given by

$$\mathcal{F}_4\boldsymbol{y}_i = \sum_{j=1}^{6}(\alpha_i - \lambda_j)k_{ij}\boldsymbol{x}_j \tag{83}$$

Because the eigenvectors of $\bar{\mathcal{F}}_3$ are orthogonal and because $\mathcal{F}_4$ is symmetric positive semidefinite, then

$$\sum_{j=1}^{6}(\alpha_i - \lambda_j)k_{ij}^2 = \alpha_i\sum_{j=1}^{6}k_{ij}^2 - \sum_{j=1}^{6}k_{ij}^2\lambda_j = \alpha_i - \sum_{j=1}^{6}k_{ij}^2\lambda_j \geq 0 \tag{84}$$

Therefore, the following condition is true:

$$\alpha_i \geq \sum_{j=1}^{6} k_{ij}^2\lambda_j \tag{85}$$

Let $\lambda_{\min} = \min[\lambda_1 \;\; \lambda_2 \;\; \cdots \;\; \lambda_6]$. Then, from Eq. (85), $\alpha_i > \lambda_{\min}$. We know that $1/\lambda_i$ is an eigenvalue of both $\bar{\mathcal{F}}_3^{-1}$ and $P$ using three observations, and $1/\alpha_i$ is an eigenvalue of both $\bar{\mathcal{F}}_4^{-1}$ and $P$ using four observations. The eigenvalue analysis can be extended to the $N$-vector observation case and indicates that each eigenvalue of $P$ using $N$ observations is less than the maximum eigenvalue of the matrix with fewer than $N$ observations. This proves that as the number of vector observations $N$ increases, more information is provided, which again intuitively makes sense.

## Examples

Observability examples using representative geometric scenarios are shown in this section. We first consider the VISNAV system configuration, shown in Fig. 1, with the following three beacon locations:

$$X_1 = 1 \text{ m}, \qquad Y_1 = 2 \text{ m}, \qquad Z_1 = 1 \text{ m}$$

$$X_2 = 1 \text{ m}, \qquad Y_2 = 2 \text{ m}, \qquad Z_2 = 2 \text{ m}$$

$$X_3 = 1 \text{ m}, \qquad Y_3 = 2 \text{ m}, \qquad Z_3 = 3 \text{ m}$$

The variances of the measurement error processes are assumed to be equal for each observation, which subsequently do not affect the observability analysis. Therefore, all measurement error variances can be set to $\sigma_i^2 = 1$ for $i = 1, 2, 3$. Also, the focal length can be set to $f = 1$ without loss in generality. The true vehicle motion is given by $X_c = 30\exp[-(1/300)t]$ m, $Y_c = 30 - (30/1800)t$ m, and $Z_c = 10 - (10/1800)t$ m. A 1800-s simulation has been performed to generate the Fisher information matrix, that is, the inverse of the covariance matrix in Eq. (10). A plot of the eigenvalues of the Fisher information at each time is shown in Fig. 3. Two of the eigenvalues are nearly equal. (The top line in the plot represents these eigenvalues.) For this example, the Fisher information matrix is clearly rank deficient. Thus, this configuration leads to an unobservable system. This is because the endpoints of the position vectors are connected by a straight line, as discussed earlier.

For the second example, we consider the following three beacon locations:

$$X_1 = 0.5 \text{ m}, \qquad Y_1 = 0.5 \text{ m}, \qquad Z_1 = 0.0 \text{ m}$$

$$X_2 = 0.5 \text{ m}, \qquad Y_2 = -0.5 \text{ m}, \qquad Z_2 = 0.0 \text{ m}$$

$$X_3 = 0.2 \text{ m}, \qquad Y_3 = 0.0 \text{ m}, \qquad Z_3 = 0.1 \text{ m}$$

A plot of the eigenvalues of the Fisher information at each time is shown in Fig. 4. Once again, two of the eigenvalues are nearly equal. (The top line in the plot represents these eigenvalues.) For this example, the Fisher information matrix is now full rank at all times. Thus, this configuration leads to an observable system. A measure of the performance in the estimation algorithm is given by the condition number (the ratio of the largest eigenvalue of the information matrix to the smallest eigenvalue). For this example, the performance improves as the vehicle approaches the beacons because they now more completely span the focal place area. However, as the vehicle moves past the beacons, the performance degrades, which is more clearly seen in Fig. 3. This is directly related to the variances of the attitude and position estimation errors. (See Ref. 3 for more details.) These examples indicate that the analysis shown in this paper can help researchers to understand and assess the observability of

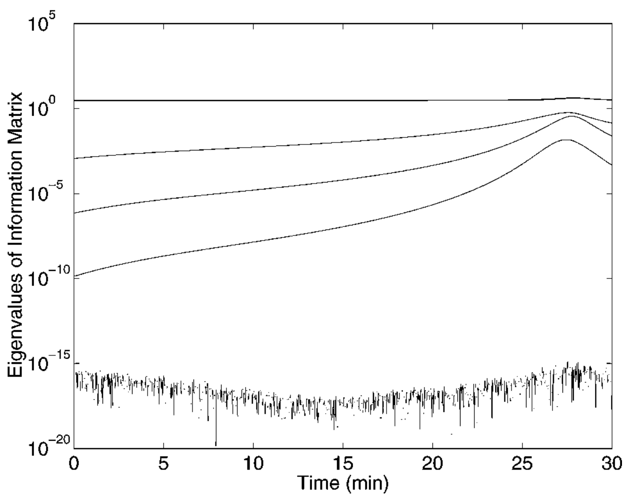


**Fig. 3 Eigenvalues of the information matrix for an unobservable case.**

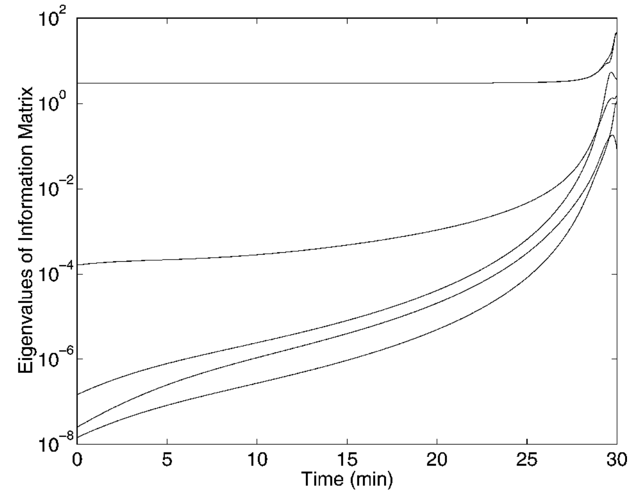


**Fig. 4 Eigenvalues of the information matrix for an observable case.**

the estimation process when using LOS measurements to determine attitude and position.

## Conclusions

An observability analysis for six-degree-of-freedom state determination using vector observations was performed. The observability analysis proved that when one vector observation is used, two pieces of information can be inferred. However, the observable quantities involve a combination of position and attitude information, which cannot be decoupled. When two-vector observations are used, the rank of the covariance matrix is four. However, only one axis of attitude and one axis of position can be determined physically, whereas the other two pieces of information involve coupled attitude/position information. When three or more vector observations are used, the covariance matrix has full rank in most cases, and a unique solution for attitude and position exists for four or more vector observations. Finally, a trace and eigenvalue analysis of the covariance matrix indicated that as the number of vector observations increases, more accurate attitude and position information is provided in general.

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